Note Title 5/4/2009

Note that the Cauchy criterion allows us to determine whether the limit exists without knowing its value.

Lemma: Let $\{X_n\}$ be a sequence of random variables with $EX_n^2 < \infty$, $\forall n$. Then there exists a random variable X such that $X_n \xrightarrow{m.s.} X$ if & only if the limit

$\lim_{m,n\to\infty} E(X_n X_m)$ exists and is finite. Furthermore,

if $X_n \xrightarrow{m.s.} X$, then $\lim_{m,n\to\infty} [X_n X_m] = EX^2$.

Proof: if $\lim_{m,n\to\infty} E(X_m X_n) = c$ for some c, then.

$$E(X_n - X_m)^2 = EX_n^2 - 2E(X_m X_n) + EX_m^2$$

$$\to c - 2c + c = 0$$

$\Rightarrow X_n$ is Cauchy in the m.s. sense $\Rightarrow X_n \xrightarrow{m.s.} X$ for some X.

Next, suppose $X_n \xrightarrow{m.s.} X$. Then

$$E(X_m X_n) = E\left(X^2 + (X_m - X)X + X(X_n - X) + (X_m - X)(X_n - X)\right)$$

By Cauchy-Schwarz inequality,

$$E|(X_n - X)X| \le \sqrt{E|X_n - X|^2 EX^2} \to 0$$

$$E|(X_m - X)(X_n - X)| \le \sqrt{E|X_n - X|^2 E|X_m - X|^2} \to 0$$

$$E|(X_m - X)X| \le \sqrt{E(X_m - X)^2 EX^2} \to 0$$

$\therefore \quad E(X_m X_n) \to EX^2.$ ⊗

We apply this to the problem of checking whether the m.s.-derivative exists.

Theorem: A process $X(t)$ with autocorrelation $R_{XX}(t_1, t_2)$ has a m.s.-derivative at time iff $\frac{\partial^2}{\partial t_1 \partial t_2} R_{XX}(t_1, t_2)$ exists at $t_1 = t_2 = t$.

Proof: By previous lemma this only depends on the existence of

$$E\left\{\left|\frac{X(t+\varepsilon_1)-X(t)}{\varepsilon_1} - \frac{X(t+\varepsilon_2)-X(t)}{\varepsilon_2}\right|^2\right\}$$

$$= E\left|\frac{X(t+\varepsilon_1)-X(t)}{\varepsilon_1}\right|^2 + E\left|\frac{X(t+\varepsilon_2)-X(t)}{\varepsilon_2}\right|^2$$

$$- 2E\left(\left|\left(\frac{X(t+\varepsilon_1)-X(t)}{\varepsilon_1}\right)\left(\frac{X(t+\varepsilon_2)-X(t)}{\varepsilon_2}\right)\right|\right) \quad \text{— (1)}$$

Now, note that

$$E\left|\frac{X(t+\varepsilon)-X(t)}{\varepsilon}\right|^2 = \frac{R_{XX}(t+\varepsilon, t+\varepsilon) - R_{XX}(t, t+\varepsilon) - R_{XX}(t+\varepsilon, t) + R_{XX}(t,t)}{\varepsilon^2}$$

$$\longrightarrow \frac{\partial^2}{\partial t_1 \partial t_2} R_{XX}(t_1, t_2) \Big|_{t_1 = t,\ t_2 = t}$$

To see this note that for a function of two variables $f(x_1, x_2)$, we have

$$\frac{\partial}{\partial x_2} f(x_1, x_2) = \lim_{h_2 \to 0} \frac{f(x_1, x_2 + h_2) - f(x_1, x_2)}{h_2}$$

and
$$\frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2) = \lim_{h_1 \to 0} \lim_{h_2 \to 0} \left[\frac{f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2)}{h_1 h_2} - \frac{f(x_1, x_2 + h_2) - f(x_1, x_2)}{h_1 h_2} \right]$$

$$= \lim_{h_1 \to 0} \lim_{h_2 \to 0} \left[\frac{f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) - f(x_1, x_2 + h_2) + f(x_1, x_2)}{h_1 h_2} \right]$$

Now comparing this expression with the one above we have the required result.

$\therefore$ in equation ① above, we have that the first two terms converge to

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_{XX}(t_1, t_2) \Big|_{t_1 = t,\ t_2 = t}$$

In a similar manner, the third term converges to

$$-2 \frac{\partial^2}{\partial t_1 \partial t_2} R_{XX}(t_1, t_2) \Big|_{t_1 = t,\, t_2 = t}$$

$\Rightarrow$ equation ① $\longrightarrow 0$ as $\varepsilon_1 \to 0$ & $\varepsilon_2 \to 0$.

For this to happen we need that R_{XX}, $\frac{\partial}{\partial t_2} R_{XX}$ and $\frac{\partial^2}{\partial t_1 \partial t_2} R_{XX}$ exist and are continuous. (* from multivariate calculus) $\boxtimes$

If $X(t)$ is WSS, then $R_{XX}(s,t) = R_{XX}(s-t) = R_{XX}(\tau)$ where $\tau = s-t$.

Suppose $R_{XX}(\tau)$, $R_{XX}'(\tau)$, $R_{XX}''(\tau)$ exist and are continuous. This is sufficient for existence of the m.s.-derivative

$$\frac{\partial}{\partial s} R_{XX}(s,t) = \frac{\partial}{\partial s} R_{XX}(s-t) = R_{XX}'(\tau)$$

$$\frac{\partial}{\partial t \partial s} R_{XX}(s,t) = -R_{XX}''(\tau).$$

A necessary condition for a WSS process to be differentiable is the following.

$$E\left[\left(\frac{X(t) - X(0)}{t}\right)^2\right] = \frac{-2\left(R_{XX}(t) - R_{XX}(0)\right)}{t^2}$$

$$\approx -\frac{2}{t} R_{XX}'(0) \text{ for small } t$$

$\therefore$ $R'_{XX}(0)$ needs to be zero for finiteness of the limit.

i.e. in addition to existence of $R'_{XX}(t)$, it needs to be zero at $t=0$.

Given the existence of the m.s.-derivative we can ask additional questions about $X'(t)$.

(i) $$E(X'(t)) = E\left(\frac{d}{dt}X(t)\right)$$

$$= \frac{d}{dt}E(X(t)) = \mu'_X(t)$$

We now justify the switch of expectation & $\frac{d}{dt}$.

Assume that the m.s. derivative exists and consider

$$A_n \triangleq \frac{X(t+\frac{1}{n}) - X(t)}{\frac{1}{n}}$$

Now, $\mu'_X(t) = \lim_{n\to\infty} E(A_n)$, and

$$\left(E(X'(t) - A_n)\right)^2 \le E|X'(t) - A_n|^2$$

Recall that $X'(t) = \lim_{\varepsilon\to 0} \text{m.s.} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$

$\therefore$ we have $E|X'(t) - A_n|^2 \longrightarrow 0$ as $n\to\infty$

$\therefore$ $EX'(t) = \mu'_X(t)$.

(ii) $R_{X'X'}(t_1, t_2) = E\left(X'(t)\big|_{t=t_1} \; X'(t)\big|_{t=t_2} \right)$

$$= E\left(\underbrace{\lim_{s_1 \to t_1} \frac{X(s_1) - X(t_1)}{s_1 - t_1}}_{\xrightarrow{m.s.} X'(t)|_{t=t_1}} \; \underbrace{\lim_{s_2 \to t_2} \frac{X(s_2) - X(t_2)}{s_2 - t_2}}_{\xrightarrow{m.s.} X'(t)|_{t=2}} \right)$$

$$= \lim_{\substack{s_1 \to t_1 \\ s_2 \to t_2}} E\left[\frac{X(s_1) - X(t_1)}{s_1 - t_1} \cdot \frac{X(s_2) - X(t_2)}{s_2 - t_2} \right]$$

$$\longrightarrow \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2}$$

i.e. the mixed partial derivative evaluated at (t_1, t_2).

In practice, we often have to work with processes that do not have m.s.-derivatives. However, we get around this difficulty by working with delta functions.

e.g. Consider $R_{XX}(t_1, t_2) = \sigma^2 e^{-\alpha|t_1 - t_2|}$ which is a common example of an auto-correlation function.

$$\frac{\partial}{\partial t_2} R_{XX}(t_1, t_2) = \begin{cases} -\alpha \sigma^2 e^{-\alpha(t_2 - t_1)} & \text{if } t_1 < t_2 \\ \alpha \sigma^2 e^{-\alpha(t_1 - t_2)} & \text{otherwise} \end{cases}$$

clearly at $t_1 = t_2$, there is a discontinuity at the above functions, and we cannot differentiate it wrt to t_1.

However we say that

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_{xx}(t_1, t_2) = 2\alpha\sigma^2 \delta(t_1 - t_2) - \alpha^2\sigma^2 e^{-\alpha|t_1 - t_2|}$$

where $\delta(t_1 - t_2)$ is the Dirac delta function.
Note that the difference between the values at the discontinuity is precisely the area of the delta function.

Next, we try to interpret the delta function.
Consider the Wiener process $W(t)$, such that $R_{WW}(t_1, t_2) = \alpha \min(t_1, t_2)$, $\alpha > 0$.

Then it is easy to see that $\frac{\partial R_{xx}}{\partial t_1 \partial t_2} = \alpha\,\delta(t_1 - t_2)$.
i.e. the derivative of the Wiener process has an auto-correlation function that is a delta function.
Such a process is called a white Gaussian noise (WGN) process in electrical engineering.