# EE523 Lecture Notes 1

Note Title 1/15/2009

**Defn.: Sample Space**

The set of all possible outcomes of an experiment is called the sample space, usually denoted by $\Omega$.

E.g. Coin tosses – $\Omega = \{H, T\}$.

**Defn.: Event**

An event is a subset of the sample space.

E.g. Let the sample space be two coin tosses i.e. $\Omega = \{HH, HT, TH, TT\}$.
then, the event that at least one head occurred is $A = \{HH, HT, TH\} \subset \Omega$.

**Question:** In general can all subsets of $\Omega$ be events.
– The answer is a bit counter-intuitive.
In general when $\Omega$ is an infinite set, not all subsets of it can be events.

**Defn.: Field**

Let $\mathcal{F} \subset 2^{\Omega}$. Then $\mathcal{F}$ forms a field if

(a) If $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

(b) If $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

(c) $\emptyset \in \mathcal{F}$.

Note that the above property implies that if $A_1, A_2, A_3, \ldots A_n \in \mathcal{F}$ for a fixed $n$

then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$.

i.e. a field is closed under finite unions.

<u>Defn.</u>: $\sigma$-field

Let $\mathcal{F} \subset 2^{\Omega}$. Then $\mathcal{F}$ is a $\sigma$-field if

(a) $\emptyset \in \mathcal{F}$.

(b) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

(c) if $A_1, A_2 \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

<u>Defn.</u>: Prob. measure

A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0,1]$ satisfying

(a) $P(\emptyset) = 0$, $P(\Omega) = 1$.

(b) If $A_1, A_2, \ldots$ is a collection of disjoint sets in $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a prob. space.

Lemma

(a) $P(A^c) = 1 - P(A)$

- $P(A \cup A^c) = P(\Omega) = 1$ & $A, A^c$ are disjoint

$\Rightarrow P(A) + P(A^c) = 1 \Rightarrow P(A^c) = 1 - P(A)$.

(b) If $A \subseteq B$, then $P(B) \geq P(A)$.

- $B = A \cup (B \setminus A) \Rightarrow P(B) = P(A) + P(B \setminus A)$
$\geq P(A)$

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

- $P(A \cup B) = P(A) + P(B \setminus A)$.
$= P(A) + P(B \setminus A \cap B)$

Now $A \cap B \subseteq B$.

$\Rightarrow P(B \setminus A \cap B) = P(B) - P(A \cap B)$.

Now we consider a technical issue. Using only the properties that we defined earlier, we show that P is a continuous set function.

<u>Lemma:</u> Let $A_1, A_2, \ldots$ be an increasing sequence of events

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \text{ and write}$$

$$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i$$

Then $P(A) = \lim_{i \to \infty} P(A_i)$.

<u>Remark:</u> The lemma is ~~stating~~ that

$$P\left(\lim_{i \to \infty} A_i\right) = \lim_{i \to \infty} P(A_i).$$

i.e. the order of limits & probability function can be interchanged.

Similarly if $B_1 \supseteq B_2 \supseteq \ldots$ then

$$B = \bigcap_{i=1}^{\infty} B_i = \lim_{i \to \infty} B_i \text{ satisfies}$$

$$P(B) = \lim_{i \to \infty} P(B_i).$$

Proof: Note that this statement is non-trivial since in general for a function $f$.

$$\lim_{i\to\infty} f(x_i) \neq f\left(\lim_{i\to\infty} x_i\right).$$

Now.

$$A = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \ldots\ldots$$

(disjoint union)

$$\Rightarrow P(A) = P(A_1) + \sum_{i=1}^{\infty} P(A_{i+1} \setminus A_i).$$

$$= P(A_1) + \sum_{i=1}^{\infty} P(A_{i+1}) - P(A_i)$$

$$= \lim_{n\to\infty} P(A_n).$$

# Conditional Probability

**Example**: $A = \{$it rains tomorrow$\}$

$B = \{$Cyride Bus is on time$\}$.

What is the chance that the bus is on time given that it rains tomorrow.

**Defn.** If $P(B) > 0$, then the conditional prob. of $A$ occurring given that $B$ occurred is defined to be,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

**Example**: Two fair dice are thrown. Given that the first shows 3, what is the probability that the total exceeds 6.

$$\Omega = \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\}.$$

$$B = \{(3, b) : 1 \leq b \leq 6\}$$

$$A = \{(a, b) : a+b > 6\}.$$

$$A \cap B = \{(3,4), (3,5), (3,6)\}.$$

$$\therefore P(A|B) = \frac{3/36}{1/6} = \frac{1}{2}.$$

<u>Lemma</u>: For any events $B_1, B_2 \dots B_n$ that are a partition of $\Omega$.

i.e. $B_i \cap B_j = \emptyset$ & $\bigcup_{i=1}^{n} B_i = \Omega$, we have that

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i)$$

<u>Proof</u>: Shall only develop the proof for $n=2$ as it extends naturally.

$$P(A) = P(A \cap B \;\dot\cup\; A \cap B^c) \quad \text{(disjoint union)}.$$

$$= P(A \cap B) + P(A \cap B^c)$$
$$= P(A \mid B) P(B) + P(A \mid B^c) P(B^c).$$

<u>Independence</u>

In general when event $B$ occurs, it changes the prob. of $A$ occurring. i.e.

$$P(A \mid B) \neq P(A).$$

However, if $P(A \mid B) = P(A)$ we say that the events $A$ & $B$ are independent. This condition is usually written as

$$P(A \cap B) = P(A) P(B)$$

More generally, a family of events $\{A_i\}_{i \in I}$ is called independent if $P\left(\bigcap_{i\in J} A_i\right) = \prod_{i\in J} P(A_i)$ for all finite subsets of $I$.

## Random Variables

Sometimes instead of the experiments, we are interested in functions of the outcome e.g.

$$\Omega = \{HH, HT, TH, TT\}.$$

& let $X(\omega) = \#$ of heads in $\omega$.

Then $X(HH)=2$, $X(HT)=1$, $X(TH)=1$, $X(TT)=0$. This may be the quantity of interest if we are betting on the number of heads.

Defn.: A random variable is a function $X : \Omega \longrightarrow \mathbb{R}$ with the property that $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for each $x \in \mathbb{R}$. Such a function is called $\mathcal{F}$-measurable

Remark: The conditions on $X$ is rather technical. In fact almost all functions encountered in engineering satisfy it.

**Defn.** Distribution Function

The distribution function of a r.v. $X$ is the function $F: \mathbb{R} \to [0,1]$, $F(x) = P(X \leq x)$.

Note that
$$P(X \leq x) = P(\{\omega \in \Omega : X(\omega) \leq x\}).$$