Iowa State University Lecturer: Aditya Ramamoorthy and Zhengdao Wang
EE 520: Topics in Communications: Multi-User Information Theory
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Lecture 12

In this lecture, we start studying the broadcast channel.

1 Preliminaries of Broadcast Channel

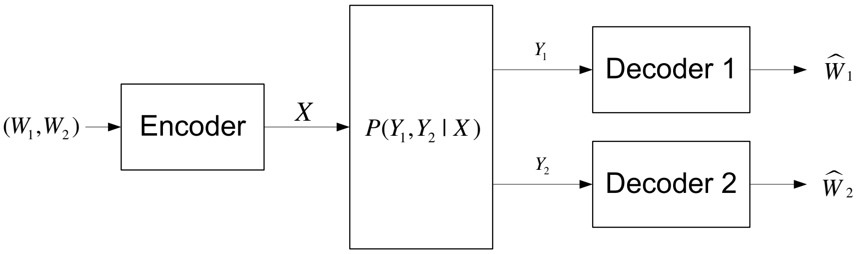

Figure 1: A two-user broadcast channel.

Fig. 1 is a schematic display of a broadcast channel. We have message W_1, W_2 to be transmitted to User 1 and 2 respectively, the encoder encodes the messages into codewords x^n which is the input of the broadcast channel $p(y_1^n, y_2^n|x^n)$. If the channel is *memoryless*, $p(y_1^n, y_2^n|x^n) = \prod_{i=1}^n p(y_{1i}, y_{2i}|x_i)$. We are only interested in the memoryless channel here.

The message $W_i \in \{1, 2, \cdots, 2^{nR_i}\}, i = 1, 2$, and a $((2^{nR_1}, 2^{nR_2}), n)$ code for a broadcast channel with independent information consists of an encoder $X : (\{1, 2, \cdots, 2^{nR_1}\} \times \{1, 2, \cdots, 2^{nR_2}\}) \to \mathcal{X}^n$ and two decoders $g_1 : \mathcal{Y}_1^n \to \{1, 2, \cdots, 2^{nR_1}\}$, $g_2 : \mathcal{Y}_2^n \to \{1, 2, \cdots, 2^{nR_2}\}$.

Definition 1. *The probability of error, achievable rate pair and capacity region are defined as:*

- *The probability of error* $P_e^{(n)} = P(\hat{W}_1 \neq W_1 \text{ or } \hat{W}_2 \neq W_2)$*, and the individual probability of error* $P_{i,e}^{(n)} = P(\hat{W}_i \neq W_i), i = 1, 2$.
- *A rate pair* (R_1, R_2) *is called achievable if* $\exists((2^{nR_1}, 2^{nR_2}), n)$ *code s.t.* $P_e^{(n)} \to 0$ *as* $n \to \infty$.
- *The capacity region is defined as the closure of all achievable* (R_1, R_2) *pairs.*

Observation 1. *The way of how the error probability is defined does not actually matter because when* $P_e^{(n)}$ *goes to zero, the individual error probabilities* $P_{i,e}^{(n)}$ *must go to zero and vice versa. This is because:*

$$P_e^{(n)} \leq P_{1,e}^{(n)} + P_{2,e}^{(n)}, \tag{1}$$

$$P_e^{(n)} \geq P_{i,e}^{(n)}, i = 1, 2. \tag{2}$$

Lemma 1. *The capacity region of the broadcast channel depends only on the conditional marginal distributions* $p(y_i|x), i = 1, 2$.

proof: By observation 1, we can equivalently use individual probability of error, and any rate pair that makes $P_e^{(n)}$ go to zero will equivalently make $P_{i,e}^{(n)}$ go to zero. Notice that decoder 1 only has access to y_1^n, and decoder 2 only has access to y_2^n. Therefore, the performance of decoder i (whether $P_{i,e}^{(n)} = P(g_i(Y_i^n) \neq W_i)$ goes to zero or not) only depends on the marginal distribution $p(y_i|x)$. Hence, we have the claim.

Another case in the broadcast channel is that there is a common information W_0 to be transmitted to both user 1,2 in addition to the individual messages W_1, W_2. In this case, we need to construct a $((2^{nR_0}, 2^{nR_1}, 2^{nR_2}), n)$ code and analyze the rate triplet (R_0, R_1, R_2). The definitions of error probability, achievable rate pairs and capacity region are simply analogs of definition 1. An interesting special case is that we have only common message to transmit.

Lemma 2. *When there is only common message to transmit through the broadcast channel the capacity is given by* $R_1 = 0, R_2 = 0$ *and*

$$R_0 < \max_{p(x)}[\min(I(X;Y_1), I(X;Y_2))]. \tag{3}$$

Observation 2. *Let* $C_i = \max_{p(x)} I(X;Y_i)$ *be the channel capacity of point-to-point channel* $p(y_i|x), (i = 1, 2)$. *The capacity for the broadcast channel with common message only is actually small than the smaller capacity of the two individual channels, i.e.*

$$\max_{p(x)}[\min(I(X;Y_1), I(X;Y_2))] \leq \min(C_1, C_2). \tag{4}$$

This is because C_1 *and* C_2 *can be achieved by different* $p(x)$*'s.*

Proof of lemma 2:

Achievability: The achievability part of the lemma is simple. Let $p^*(x)$ be the distribution that maximize (3). Generate 2^{nR_0} codewords according to $p^*(x)$. By the choice of $p^*(x)$, we are allowed to choose rate R_0 arbitrarily close to (3). But by (4), we know $R_0 < C_i, i = 1, 2$. That means for individual channels, we still transmit below the individual capacity. By the coding theorem of point to point channel, we know with rate such a rate R_0, we are able to reliably transmit on both channel 1 and 2, i.e. $P_{i,e}^{(n)} \to 0$ as $n \to \infty$, and hence by observation 1 $P_e^{(n)} \to 0$. Therefore R_0 is achievable.

Converse:

$$\begin{aligned}
nR_0 &\leq H(W_0) \tag{5}\\
&= H(W_0) - H(W_0|Y_1^n) + H(W_0|Y_1^n) \tag{6}\\
&= I(W_0; Y_1^n) + n\epsilon \tag{7}\\
&= H(Y_1^n) - H(Y_1^n|W_0) + n\epsilon \tag{8}\\
&\leq H(Y_1^n) - H(Y_1^n|X^n) + n\epsilon \tag{9}\\
&\leq \sum_{i=1}^{n} H(Y_{1i}) - H(Y_{1i}|X_i) + n\epsilon \tag{10}\\
&= \sum_{i=1}^{n} I(Y_{1i}; X_i) + n\epsilon \tag{11}\\
&= n(\frac{1}{n}\sum_{i=1}^{n} I(Y_{1i}; X_i)) + n\epsilon \tag{12}\\
&= n(\frac{1}{n}\sum_{i=1}^{n} I(p(x_i), p(y_1|x)) + n\epsilon \tag{13}\\
&\leq nI(\frac{1}{n}\sum_{i=1}^{n} p(x_i), p(y_1|x)) + n\epsilon \tag{14}\\
&= nI(X; Y_1) + n\epsilon, \text{ where } X \text{ is distributed as } \tfrac{1}{n}\textstyle\sum_{i=1}^{n} p(x_i). \tag{15}
\end{aligned}$$

(7) is due to Fano's inequality, (9) follows from the Markov property, and (14) uses the fact that mutual information is concave in $p(x)$. Let $\epsilon \to 0$, we have

$$R_0 \leq I(X; Y_1). \tag{16}$$

And similarly,

$$R_0 \leq I(X; Y_2), \tag{17}$$

with the same X. Combining the two conditions,

$$\begin{aligned}
R_0 &\leq \min(I(X; Y_2), I(X; Y_2)) \tag{18}\\
&\leq \max_{p(x)}[\min(I(X; Y_1), I(X; Y_2))]. \tag{19}
\end{aligned}$$