Continuous-time F.T.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{(freq-domain)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega. \qquad \text{(time-domain)}$$

Note if you substitute $\omega = 2\pi f$, then we obtain

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

For periodic signal.

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$x_{T_0}(t) = \sum_{-\infty}^{\infty} a_k\, e^{jk\omega_0 t}.$$

E.g. $x(t) = e^{-7t} u(t)$

$$\Rightarrow X(\omega) = \int_{0}^{\infty} e^{-7t}\, e^{-j\omega t}\, dt$$

$$= \frac{1}{7+j\omega} \longleftrightarrow e^{-7t} u(t)$$

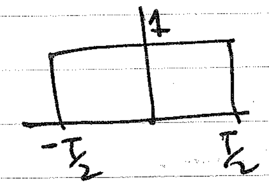
Usually ~~we~~ we use F.T. pairs, instead of actually computing integrals
Note : ~~the~~ FT representation is unique.

Sufficient condition for existence of F.T. : $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

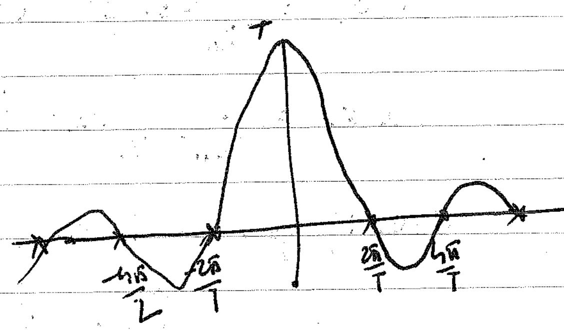
however many signals in practice do not follow it.

Some important F.T. pairs

$$e^{-at} u(t) \longleftrightarrow \frac{1}{a+j\omega} \quad \text{for } a>0$$

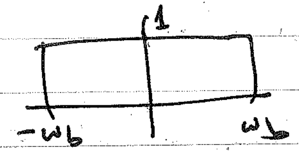


$$\longleftrightarrow \quad \frac{\sin(\omega T/2)}{\omega/2}$$



Bandwidth of a signal is the smallest ω_0 s.t. $|X(\omega)| = 0, \ \forall \omega \geqslant \omega_0$.

$$\frac{\sin(\omega_b t)}{\pi t} \longleftrightarrow$$



$$A\,\delta(t) \longleftrightarrow A$$

$$1 \longleftrightarrow 2\pi\,\delta(\omega)$$

$$A\cos(\omega_0 t + \phi) \longleftrightarrow \pi A e^{j\phi}\delta(\omega - \omega_0) + \pi A e^{-j\phi}\delta(\omega + \omega_0)$$

F.T. of periodic signals

$x(t)$ can be represented as

$$x(t) = \sum a_k e^{jk\omega_0 t}$$

$$\Rightarrow X(\omega) = \sum_{-\infty}^{\infty} 2\pi a_k \,\delta(\omega - k\omega_0)$$

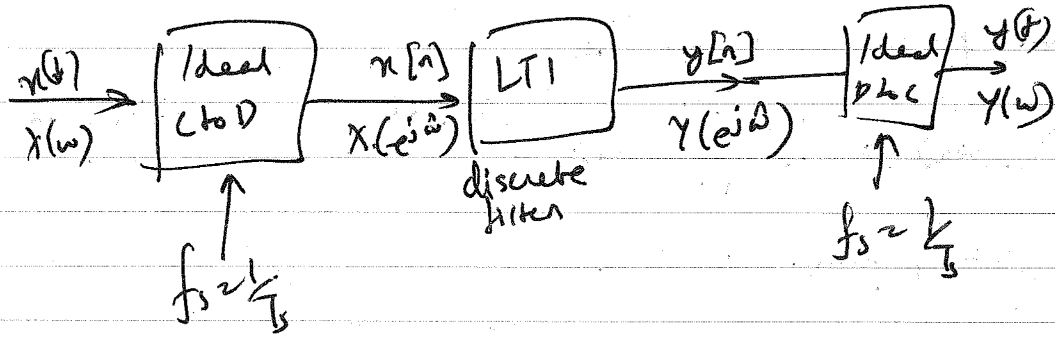
Likewise inverse DTFT.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})\, e^{j\hat{\omega} n}\, d\hat{\omega}.$$

~~Should I talk~~

Filtering CT signals using discrete operations]

$X(e^{j\hat{\omega}})$ is periodic with period 2π. since $e^{j2\pi}=1$.



$$Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})\, X(e^{j\hat{\omega}})$$

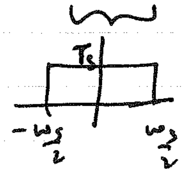
$$y(t) = \sum_{-\infty}^{\infty} y[n]\, h_r(t - nT_s)$$

$$Y(\omega) = \sum_{-\infty}^{\infty} y[n]\, H_r(\omega)\, e^{-j\omega n T_s}$$

$$= H_r(\omega)\, Y(e^{j\omega T_s}).$$

$$= H_r(\omega)\, H(e^{j\omega T_s})\, X(e^{j\omega T_s})$$

$$= \underbrace{H_r(\omega)} H(e^{j\omega T_s}) \cdot \underbrace{\frac{1}{T_s}\sum_{-\infty}^{\infty} X_c\left(\omega - \frac{2\pi}{T_s}k\right)}_{\text{sum of shifted/scaled copies of } X_c(\omega)}$$



Discrete Fourier Transform

$$X_c(\omega) = \int x_c(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow \hat{X}_c(\omega) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\, e^{-j\omega n T_s} \cdot T_s \qquad \text{(approx. to CTFT).}$$

* However ω is still continuous.
* Limits are $-\infty$ to $+\infty$.

Another approx. at freq. ω_k

$$X_c(\omega_k) = T_s \sum_{n=0}^{L-1} x[n]\, e^{-j\omega_k n T_s} \qquad \text{(For large enough } L\text{).}$$

What set of freq. to use?

$$\omega_k = \frac{2\pi}{T_s} \cdot \left(\frac{k}{N}\right) \qquad , \; k = 0, 1, \ldots, N-1.$$

$\therefore$ we have.

$$\underbrace{\frac{1}{T_s} \hat{X}_c\left(\frac{2\pi}{T_s} \cdot \frac{k}{N}\right)}_{\text{approx. to } \frac{1}{T_s} X(\omega_k).} = \sum_{n=0}^{L-1} x[n]\, e^{-j \frac{2\pi}{T_s} \frac{k}{N} n T_s}$$

$$= \sum_{n=0}^{L-1} x[n]\, e^{-j 2\pi \frac{k}{N} n}.$$

DFT.

$$X[k] = \sum_{n=0}^{L-1} x[n]\, e^{-j 2\pi \frac{k}{N} n},$$

$$\sum_{-\infty}^{\infty} \delta(t-nT_s) \longleftrightarrow \frac{2\pi}{T_s} \sum \delta(\omega-k\omega_s) \qquad \left(\omega_s=\frac{2\pi}{T_s}\right).$$

Various properties under different operations – consult 2nd text.

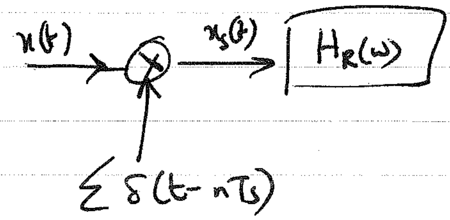
Important

⇔ Convolution in time ⟺ Multiplication in F.T.

$$x(t) * h(t) \longleftrightarrow X(\omega)\, H(\omega).$$

$$x(t)\, h(t) \longleftrightarrow \frac{1}{2\pi} X(\omega) * H(\omega).$$

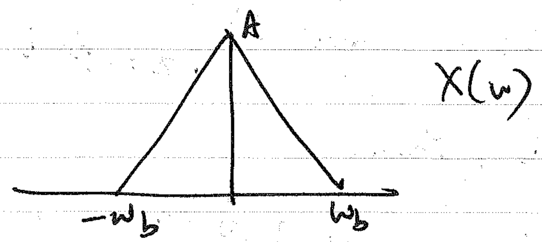
<u>Sampling theorem</u>

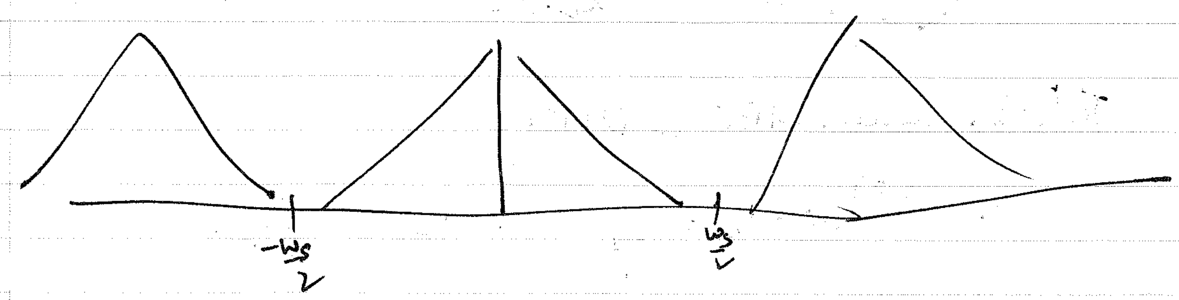


$$x_s(t) = \sum x(nT_s)\, \delta(t-nT_s)$$

$$X_s(j\omega) = \frac{1}{2\pi} X(\omega) * \sum \frac{2\pi}{T_s} \delta(\omega-k\omega_s)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega-k\omega_s).$$





Interpolation formula.

$$x_r(t) = \sum_{-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t-nT_s)\right]}{\frac{\pi}{T_s}(t-nT_s)}$$

In practice we don't need to implement an impulse train.
Usually we only have a discrete sequence $x[n]$ available to us.

$$x_s(t) = x_c(t) \sum \delta(t-nT_s)$$

$$= x_c(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_0 t}$$

$$= \sum x(nT_s)\,\delta(t-nT_s)$$

Taking F.T. on both sides.

$$\sum_{-\infty}^{\infty} x[n]\, e^{-j\omega n T_s} = \frac{1}{T_s} \sum_{-\infty}^{\infty} X_c\left(\omega - \frac{2\pi}{T_s}k\right)$$

LHS is precisely $X(z) = \sum x[n]\, z^{-n} \Big|_{z=e^{+j\omega T_s}}$

Define $\hat{\omega} = \omega T_s$, then.

$$X(e^{j\hat{\omega}}) = \sum x[n]\, e^{-j\hat{\omega} n} \quad \text{(DTFT.)}$$

Relation between CTFT & DTFT.

$$X(e^{j\hat{\omega}})\Big|_{\hat{\omega}=\omega T_s} = \frac{1}{T_s} \sum X_c\left(\omega - \frac{2\pi}{T_s}k\right).$$

Generally, we choose $L = N$.
- inverse algo exists
- efficient algos exists for DFT computation

Note: For DTFT.

$$X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{-\infty}^{\infty} X_c(\omega - k\omega_s)$$

if ω_s is high enough to avoid aliasing,

$$X(e^{j\omega T_s}) = \frac{1}{T_s} X_c(\omega) \qquad \text{for } |\omega| \leq \frac{\omega_s}{2}.$$

$\Rightarrow$ CTFT can be computed by determining DTFT.

<u>Reln. between DTFT & DFT</u>

if $x[n] \neq 0$ only for $0 \leq n \leq L-1$, then DTFT

$$X(e^{j\hat{\omega}}) = \sum_{0}^{L-1} x[n] e^{-j\hat{\omega} n}$$

& DFT is

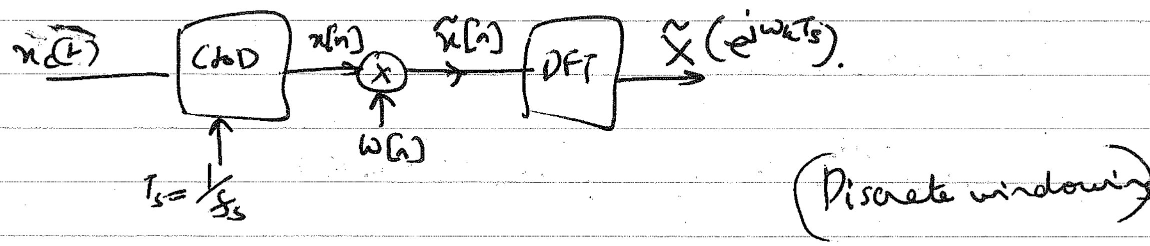
$$X[k] = \sum_{0}^{L-1} x[n] e^{-j 2\pi \frac{k}{N} n}$$

$\Rightarrow$ evaluating DTFT at $\hat{\omega} = \frac{2\pi k}{N}$, $k = 0, \ldots, N-1$.

<u>Effect of finite L</u>

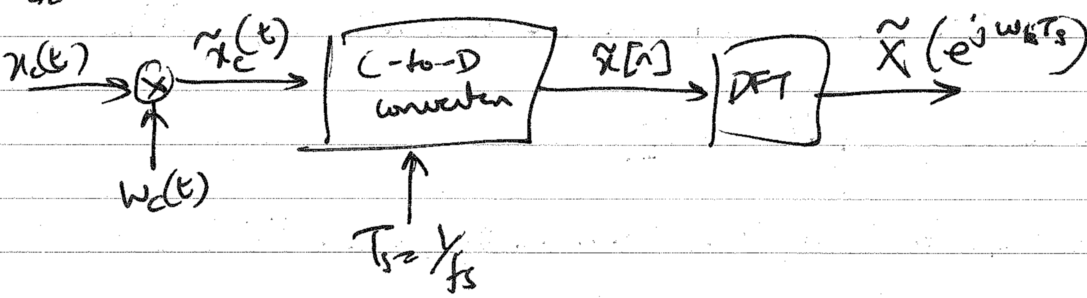
Effect of finite L can be modeled by multiplying by

$$w[n] = \begin{cases} 1 & 0 \leq n \leq L-1 \\ 0 & \text{otherwise} \end{cases}$$



$$\tilde{X}(e^{j\omega_k T_s}) = \sum w[n]\, x[n]\, e^{-j\omega_k T_s n}$$

Equivalent cont.-time windowing can be arrived at



These are equivalent if $w[n] = w_c(nT_s)$. $\forall n$.

e.g. in this case.

$$w_c(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

$$(L-1)T_s < T \leq LT_s.$$

Example

$$x_c(t) = A_0 + A_1 \cos(\omega_1 t + \phi_1).$$

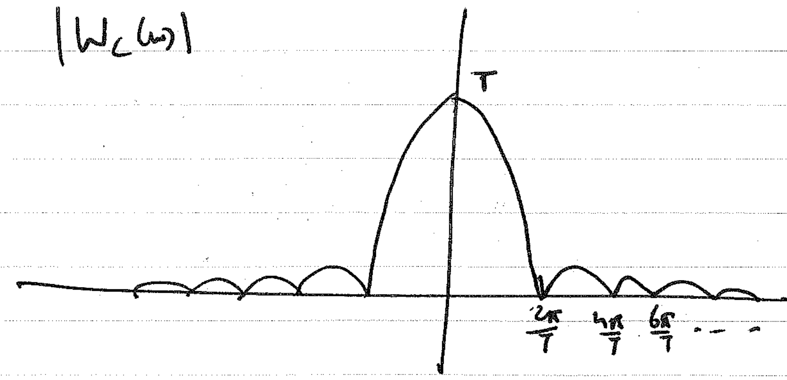
$$X_c(\omega) = 2\pi A\, \delta(\omega) + \pi A_1 e^{j\phi_1} \delta(\omega - \omega_1) + \pi A_1 e^{-j\phi_1} \delta(\omega + \omega_1)$$

$$\tilde{x}_c(t) = x_c(t)\, w_c(t)$$

$$= A_0 w_c(t) + A_1 w_c(t) \cos(\omega_1 t + \phi_1).$$

$$\Rightarrow \tilde{X}_c(\omega) = A_0 W_c(\omega) + \frac{1}{2} \overset{A_1 e^{j\phi_1}}{\tilde{X}_1} W_c(\omega - \omega_1) + \frac{1}{2} X_1^* W_c(\omega + \omega_1)$$

Now $W_c(\omega) = \left(\dfrac{\sin(\omega T/2)}{\omega/2} \right) e^{-j\omega T/2}.$



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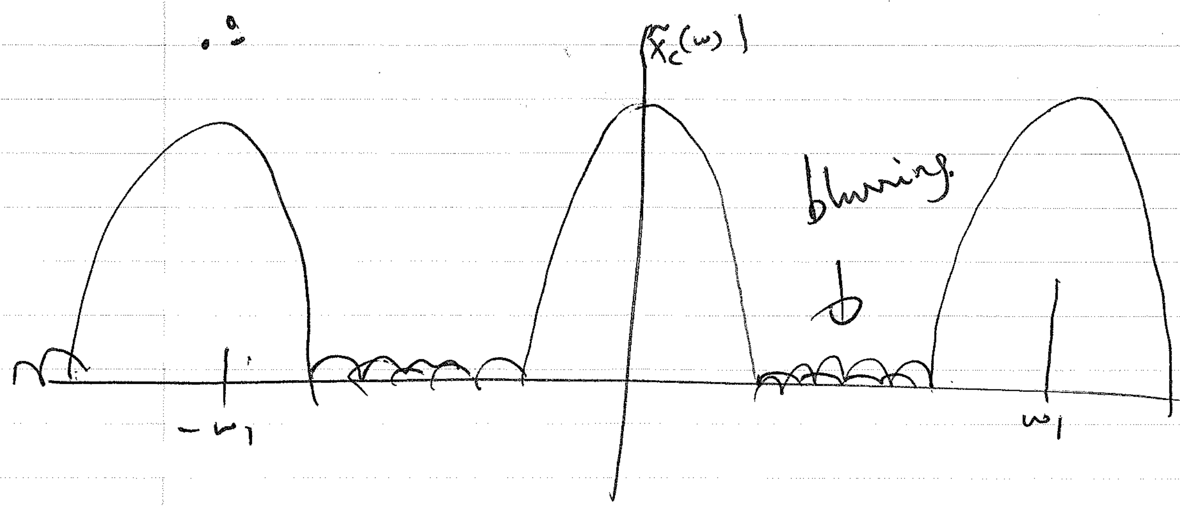

Way to reduce blurring
- Increase L or time wt. dow lengths.

DFT when L<N

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j 2\pi \frac{k}{N} n}, \quad k = 0,..,N-1$$

IDFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j 2\pi \frac{k}{N} n}$$

Dan
Adrian
dadrian
@iastate.edu.

Table 11-2: Basic Fourier transform pairs.

Table of Fourier Transform Pairs	
Time-Domain: $x(t)$	***Frequency-Domain:*** $X(j\omega)$
$e^{-at}u(t) \quad (a>0)$	$\dfrac{1}{a+j\omega}$
$e^{bt}u(-t) \quad (b>0)$	$\dfrac{1}{b-j\omega}$
$u(t+\frac{1}{2}T)-u(t-\frac{1}{2}T)$	$\dfrac{\sin(\omega T/2)}{\omega/2}$
$\dfrac{\sin(\omega_b t)}{\pi t}$	$[u(\omega+\omega_b)-u(\omega-\omega_b)]$
$\delta(t)$	1
$\delta(t-t_d)$	$e^{-j\omega t_d}$
$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$A\cos(\omega_0 t+\phi)$	$\pi Ae^{j\phi}\delta(\omega-\omega_0)+\pi Ae^{-j\phi}\delta(\omega+\omega_0)$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin(\omega_0 t)$	$-j\pi\delta(\omega-\omega_0)+j\pi\delta(\omega+\omega_0)$
$\displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega-k\omega_0)$
$\displaystyle\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$\displaystyle\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega-\frac{2\pi}{T}k)$

Table 11-3: Basic Fourier transform properties.

Table of Fourier Transform Properties		
Property Name	***Time-Domain:*** $x(t)$	***Frequency-Domain:*** $X(j\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(j\omega) + bX_2(j\omega)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time-Reversal	$x(-t)$	$X(-j\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(j(\omega/a))$
Delay	$x(t-t_d)$	$e^{-j\omega t_d}X(j\omega)$
Modulation	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Modulation	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0))$
Differentiation	$\dfrac{d^k x(t)}{dt^k}$	$(j\omega)^k X(j\omega)$
Convolution	$x(t) * h(t)$	$X(j\omega)H(j\omega)$
Multiplication	$x(t)p(t)$	$\frac{1}{2\pi}X(j\omega) * P(j\omega)$

If we apply the Fourier transform to both sides of (11.94) and use the delay property, we obtain

$$Y(j\omega) = X(j\omega) + \alpha e^{-j\omega t_d} X(j\omega)$$

Then we factor out $X(j\omega)$ to obtain

$$Y(j\omega) = \left(1 + \alpha e^{-j\omega t_d}\right) X(j\omega) \qquad (11.95)$$

Applying the convolution property of the Fourier transform to (11.95) leads us to the conclusion that the echo system is LTI and that its frequency response is given by

$$H(j\omega) = 1 + \alpha e^{-j\omega t_d} \qquad (11.96)$$